Classes of optimal low-hit-zone frequency-hopping sequence sets with new parameters

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Abstract

In order to minimize or reduce the mutual interference, low-hit-zone (LHZ) frequency-hopping sequence (FHS) sets with optimal periodic partial Hamming correlation (PPHC) properties have been well applied in quasi-synchronous (QS) frequency-hopping multiple-access (FHMA) communication systems. In this paper, via Cartesian product, two designs of LHZ-FHS sets are proposed. Besides, three classes of LHZ-FHS sets with optimal PPHC properties and new parameters not included in the related literature are obtained.

1 Introduction

Due to advantages as anti-jamming, anti-fading, multiple access and secure properties, frequency-hopping multiple-access (FHMA) systems have found wide applications

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in both military and civil communications [1, 2, 3]. Usually, if two or more users transmit information on the same frequency simultaneously, mutual interference (MI) which is mainly controlled by the Hamming correlations of the employed FHS set occurs [3, 4]. Obviously, it is desirable to maintain the MI between transmitters at a level as low as possible. Usually, the correlation window length of FHMA systems is shorter than the sequence length of the employed FHS set. Furthermore, owing to the limited synchronous time or the hardware complexity, and the fact that the correlation window length may vary along with the channel conditions at any time, the periodic partial Hamming correlation (PPHC) plays an important role in measuring the performance of FHMA systems.

In quasi-synchronous (QS) FHMA systems, some relative time delay between different users within a zone around the origin can be allowed, and the low-hit-zone (LHZ) FHS sets can be well applied in such systems [5]. Moreover, to better meet the needs of different frequency-hopping communication scenarios, it is urgent to design more LHZ-FHS sets with both desired PPHC properties and new flexible parameters.

For an LHZ-FHS set, its maximum PPHC within the LHZ together with the LHZ, the sequence length, the sequence set size, the frequency slot size and the correlation window length are limited by some mathematical formulas known as theoretical bounds. Without loss of generality, when we say the maximum PPHC of an LHZ-FHS set, we refer to that within the LHZ. The first research of the theoretical bounds on the PPHC of LHZ-FHS sets was reported by Niu *et al.* [6]. Thereafter, the authors of the present paper slightly improved the bounds in [6] and obtained few classes of LHZ-FHS sets with optimal PPHC properties [7, 8].

The remainder of this paper is organized as follows. In section 2, some relevant definitions and lower bounds on LHZ-FHS sets are introduced. In Section 3, two new designs of LHZ-FHS sets through Cartesian product are presented. In Section 4, three new classes of LHZ-FHS sets with optimal PPHC properties are proposed. Finally, some concluding remarks are given in Section 5.

2 Preliminaries

Let $F = \{f_0, f_1, \cdots, f_{r-1}\}$ be a frequency slot set with r available frequency slots, $S = \{s_0, s_1, \cdots, s_{N-1}\}$ be an FHS set with N frequency-hopping sequences of length L over F so that every element of each sequence in S is in F. For any two frequency-hopping sequences $s_i = [s_i(0), s_i(1), \cdots, s_i(L-1)], s_j = [s_j(0), s_j(1), \cdots, s_j(L-1)] \in S, 0 \le i, j < N$, and the correlation window length $W, 1 \le W \le L$ starting at $w, 0 \le w < L$, the PPHC function $H_{s_i,s_j}(w|W;\tau)$ between s_i and s_j at time delay τ is defined as follows:

$$H_{s_i,s_j}(w|W;\tau) \;=\; \sum_{x=w}^{w+W-1} h[s_i(x), s_j(x+\tau)],\; 0 \le \tau < L,$$

where $h[s_i(x), s_j(x+\tau)] = 1$ if $s_i(x) = s_j(x+\tau)$, and 0 otherwise, and all the operations among the position indices are performed modulo L. What is more, $H_{s_i,s_j}(w|W;\tau)$ is called as the periodic partial Hamming autocorrelation function if $i = j$, while it is called as the periodic partial Hamming cross-correlation function if $i \ne j$.

The maximum periodic partial Hamming autocorrelation $H_{pam}(S;W)$, the maximum periodic partial Hamming cross-correlation $H_{pcm}(S;W)$, and the maximum PPHC $H_{pm}(S;W)$ of S are respectively defined as follows:

$$\begin{aligned}&H_{pam}(S;W)=\max\{H_{s_i,s_i}(w|W;\tau),1\le\tau<L,0\le w<L,0\le i<N\},\\&H_{pcm}(S;W)=\max\{H_{s_i,s_j}(w|W;\tau),0\le\tau,w<L,0\le i\ne j<N\},\\&H_{pm}(S;W)=\max\{H_{pam}(S;W),H_{pcm}(S;W)\}.\end{aligned}$$

Let integers $H_{paz}>0, H_{pcz}>0$. Then, the periodic partial Hamming autocorrelation LHZ L_{paz}, the periodic partial Hamming cross-correlation LHZ L_{pcz}, the PPHC LHZ L_{pz}, and the maximum PPHC $H_{pzm}(S;W)$ within the LHZ of S are respectively defined as:

$$\begin{aligned}&L_{paz}=\max\{Z|H_{s_i,s_i}(w|W;\tau)\le H_{paz},0\le w<L,0\le i<N,0<\tau\le Z\},\\&L_{pcz}=\max\{Z|H_{s_i,s_j}(w|W;\tau)\le H_{pcz},0\le w<L,0\le i\ne j<N,0\le\tau\le Z\},\\&L_{pz}=\min\{L_{paz},L_{pcz}\},\\&H_{pzm}(S;W)=\max\{H_{paz},H_{pcz}\}.\end{aligned}$$

Specially, $H_{pzm}(S;W)$ can be viewed as the maximum PHC within the LHZ of S if $W=L$ and denoted as $H_{zm}(S)$, while it can be viewed as the maximum PPHC of the conventional FHS set S if $L_{pz}=L-1$ and denoted as $H_{pm}(S;W)$.

In 2010, Niu *et al.* [6] derived the following lower bounds on the maximum PPHC of an LHZ-FHS set.

Lemma 1 ([6]). *Let S be an LHZ-FHS set with N frequency-hopping sequences of length L over the frequency slot set F with size r, and L_{pz} be the LHZ of S with regard to $H_{pzm}(S;W)$. Then, for any positive integer Z, $0\le Z\le L_{pz}$, we have*

$$H_{pzm}(S;W)\ge\left\lceil\frac{(Z+1)\big[(2I+1)LN-I(I+1)r\big]-L^2N}{(NZ+N-1)LN}\cdot\frac{W}{L}\right\rceil,\tag{1}$$

where $I=\left\lfloor\frac{LN}{r}\right\rfloor$.

Recently, Zhou *et al.* [7] slightly improved the bound (1) and obtained the following conclusion.

Lemma 2 ([7]). *Let S be an LHZ-FHS set with N frequency-hopping sequences of length L over the frequency slot set F with size r, and L_{pz} be the LHZ of S with regard to $H_{pzm}(S;W)$. Then, for any positive integer Z, $0\le Z\le L_{pz}$, we have*

$$H_{pzm}(S;W)\ge\left\lceil\left\lceil\frac{(Z+1)\big[(2I+1)LN-I(I+1)r\big]-L^2N}{(NZ+N-1)LN}\right\rceil\cdot\frac{W}{L}\right\rceil.\tag{2}$$

Obviously, conventional FHS sets are special LHZ-FHS sets with LHZ values equal sequence lengths minus 1. Assume that we have an FHS set S consisting of N frequency-hopping sequences with length L over the frequency slot set F with size r and having maximum PPHC $H_{pzm}(S;W)$ within the correlation window length W and LHZ L_{pz}. For the sake of simplicity, we denote S as an $(L,N,r,W,L_{pz},H_{pzm}(S;W))$ LHZ-FHS set. Besides, if $L_{pz}=L-1$, we denote S as an $(L,N,r,W,H_{pm}(S;W))$ FHS set.

Definition 3. For the $(L, N, r, W, L_{pz}, H_{pzm}(S; W))$ LHZ-FHS set S, if its parameters meet the bound (2) with equality, it is said to have optimal PPHC property.

3 New designs of LHZ-FHS sets through Cartesian product

In [7], through Cartesian product, the authors have presented two generalized methods to construct LHZ-FHS sets and mainly concerned about the maximum PPHC of the constructed LHZ-FHS sets. In this section, we will expand the results obtained before and introduce two new designs of LHZ-FHS sets. Meanwhile, we also mainly pay our attention to the maximum PPHC of newly constructed LHZ-FHS sets.

Design 1

Step 1: Let $E^i = \left\{e^i_j = [e^i_j(x), 0 \le x < L_{e^i}], 0 \le j < N_{e^i}\right\}, 0 \le i < k$ be an $(L_{e^i}, N_{e^i}, r_{e^i}, W_{e^i}, Z_{e^i}, \left\lceil \frac{W_{e^i}}{T_{e^i}} \right\rceil)$ LHZ-FHS set over F_{e^i} with $Z_{e^0} < Z_{e^1} < \cdots < Z_{e^{k-1}}$, $T_{e^i} \mid L_{e^i}$ and satisfying

$$H_{e^i_j, e^i_u}(0) = 0\ ,\ 0 \le j \ne u < N_{e^i}$$

and

$$\gcd(T_{e^i}, T_{e^j}) = 1,\ 0 \le i \ne j < k.$$

Step 2: Let $L_d = \mathrm{lcm}(L_{e^0}, \cdots, L_{e^{k-1}})$ and $N_d = N_{e^0} N_{e^1} \cdots N_{e^{k-1}}$. Then, define a new FHS set $D = \left\{d_j = [d_j(x), 0 \le x < L_d], 0 \le j < N_d\right\}$ over $F_{e^0} \times \cdots \times F_{e^{k-1}}$ as:

$$d_j(x) = d_{j_0, j_1, \cdots, j_{k-1}}(x) = \left(e^0_{j_0}(\langle x \rangle_{L_{e^0}}), e^1_{j_1}(\langle x \rangle_{L_{e^1}}), \cdots, e^{k-1}_{j_{k-1}}(\langle x \rangle_{L_{e^{k-1}}})\right), \tag{3}$$

where $0 \le j < N_d$, $0 \le j_i < N_{e^i}, 0 \le i < k$.

Theorem 4. *Let $r_d = r_{e^0} r_{e^1} \cdots r_{e^{k-1}}$, $T_d = T_{e^0} T_{e^1} \cdots T_{e^{k-1}}$ and $Z_d = Z_{e^0}$. Then, the FHS set D formed in Design 1 is an $(L_d, N_d, r_d, W_d, Z_d, \left\lceil \frac{W_d}{T_d} \right\rceil)$ LHZ-FHS set.*

Proof. Apparently, there are N_d sequences of length L_d over $F_{e^0} \times \cdots \times F_{e^{k-1}}$ in the FHS set D. Since we have $H_{pzm}(E^i, W_{e^i}) = \left\lceil \frac{W_{e^i}}{T_{e^i}} \right\rceil$ for $0 \le \tau \le Z_{e^i}$, there is one and only one hit within the correlation window length T_{e^i}. For any $d_i, d_j \in D, 0 \le i, j < N_d$, $0 \le \tau \le Z_d$ and the correlation window length T_d starting at w, the PPHC function between d_i and

d_j at time delay τ can be written as:

$$\begin{aligned}H_{d_i,d_j}(w|T_d;\tau) &= \sum_{x=w}^{w+T_d-1} h[d_i(x),d_j(x+\tau)]\\ &= \sum_{x_0=0}^{T_{e^0}-1}\sum_{x_1=0}^{T_{e^1}-1}\cdots\sum_{x_{k-1}=0}^{T_{e^{k-1}}-1} h[e^0_{i_0}(\langle w+x^{'}\rangle_{L_{e^0}}),e^0_{j_0}(\langle w+x^{'}+\tau\rangle_{L_{e^0}})]\\ &\quad\cdot h[e^1_{i_1}(\langle w+x^{'}\rangle_{L_{e^1}}),e^1_{j_1}(\langle w+x^{'}+\tau\rangle_{L_{e^1}})]\\ &\quad\cdots h[e^{k-1}_{i_{k-1}}(\langle w+x^{'}\rangle_{L_{e^{k-1}}}),e^{k-1}_{j_{k-1}}(\langle w+x^{'}+\tau\rangle_{L_{e^{k-1}}})]\\ &= \sum_{x_0=0}^{T_{e^0}-1} h[e^0_{i_0}(\langle x_0\rangle_{L_{e^0}}),e^0_{j_0}(\langle x_0+\tau\rangle_{L_{e^0}})]\\ &\quad\cdot\sum_{x_1=0}^{T_{e^1}-1} h[e^1_{i_1}(\langle x_1\rangle_{L_{e^1}}),e^1_{j_1}(\langle x_1+\tau\rangle_{L_{e^1}})]\\ &\quad\cdots\sum_{x_{k-1}=0}^{T_{e^{k-1}}-1} h[e^{k-1}_{i_{k-1}}(\langle x_{k-1}\rangle_{L_{e^{k-1}}}),e^{k-1}_{j_{k-1}}(\langle x_{k-1}+\tau\rangle_{L_{e^{k-1}}})],\end{aligned}$$

where $x=w+x^{'}$, $x^{'}=x_0+T_{e^0}x_1+T_{e^0}T_{e^1}x_2+\cdots+T_{e^0}\cdots T_{e^{k-2}}x_{k-1}$ and the last equality holds since $T_{e^0},T_{e^1},\cdots,T_{e^{k-1}}$ are coprime integers and for all $0\le j<k-1$, $T_{e^0}\cdots T_{e^j}$, $2T_{e^0}\cdots T_{e^j}$,$\cdots$,$(T_{e^{j+1}}-1)T_{e^0}\cdots T_{e^j}$ are all distinct modulo $T_{e^{j+1}}$. As $H_{e^i_j,e^i_u}(0)=0, 0\le j\ne u<N_{e^i}$, we have

$$H_{d_i,d_j}(w|T_d;\tau)\le 1$$

for any $0<\tau\le Z_d$ when $0\le i=j<N_d$, and for any $0\le\tau\le Z_d$ when $0\le i\ne j<N_d$.

Let $W_d=t_1T_d+t_2$. Then for any $d_i,d_j\in D, 0\le i,j<N_d$, $0\le\tau\le Z_d$ and the correlation window length W_d starting at w, the PPHC function between d_i and d_j at time delay τ can be written as:

$$\begin{aligned}H_{d_i,d_j}(w|W_d;\tau) &= \sum_{x=w}^{w+W_d-1} h[d_i(x),d_j(x+\tau)]\\ &= \sum_{l=0}^{t_1-1}\sum_{x=w+lT_d}^{w+lT_d+T_d-1} h[d_i(x),d_j(x+\tau)]+\sum_{x=w+t_1T_d}^{w+t_1T_d+t_2-1} h[d_i(x),d_j(x+\tau)]\\ &\le \begin{cases} t_1, & \text{if } t_2=0\\ t_1+1, & \text{otherwise}\end{cases}.\end{aligned}$$

Thus, it is obvious that

$$H_{d_i,d_j}(w|W_d;\tau)\le\left\lceil\frac{W_d}{T_d}\right\rceil.$$

for any $0 < \tau \le Z_d$ when $0 \le i = j < N_d$, and for any $0 \le \tau \le Z_d$ when $0 \le i \ne j < N_d$. That is to say, $H_{pzm}(D; W_d) = \left\lceil \frac{W_d}{T_d} \right\rceil$ within the correlation window length W_d and LHZ Z_d. □

Design 2

Step 1: Let $E^i = \left\{ e_j^i = [e_j^i(x), 0 \le x < L_{e^i}], 0 \le j < N_{e^i} \right\}, 0 \le i < k-1$ be an $(L_{e^i}, N_{e^i}, r_{e^i}, We^i, Z_{e^i}, \left\lceil \frac{We^i}{T_{e^i}} \right\rceil)$ LHZ-FHS set over F_{e^i} with $Z_{e^0} < Z_{e^1} < \cdots < Z_{e^{k-2}}$, $T_{e^i} \mid L_{e^i}$ and satisfying

$$H_{e_j^i, e_u^i}(0) = 0 \ , \ 0 \le j \ne u < N_{e^i}$$

and

$$\gcd(T_{e^i}, T_{e^j}) = 1, \ 0 \le i \ne j < k-1.$$

Step 2: Let $G = \left\{ g_j = [g_j(x), 0 \le x < L_g], 0 \le j < N_g \right\}$ be an $(L_g, N_g, r_g, W_g, Z_g, \left\lceil \frac{W_g}{T_g} \right\rceil)$ LHZ-FHS set over F_g with $\gcd(T_g, T_{e^i}) = 1, 0 \le i < k-1$, $T_g \mid L_g$ and satisfying

$$H_{g_j, g_u}(0) \ne 0 \ , \ 0 \le j \ne u < N_g.$$

Step 3: Let $L_{d'} = \mathrm{lcm}(L_{e^0}, \cdots, L_{e^{k-2}}, L_g)$ and $N_{d'} = N_{e^0} N_{e^1} \cdots N_{e^{k-2}}$. Then, define a new FHS set $D' = \left\{ d_j' = \left[d_j'(x), 0 \le x < L_{d'} \right], 0 \le j < N_{d'} \right\}$ over $F_{e^0} \times \cdots \times F_{e^{k-2}} \times F_g$ as:

$$d_j'(x) = d_{j_0, j_1, \cdots, j_{k-2}}'(x) = \left(e_{j_0}^0(\langle x \rangle_{L_{e^0}}), \cdots, e_{j_{k-2}}^{k-2}(\langle x \rangle_{L_{e^{k-2}}}), g_{\langle j \rangle_{N_g}}(\langle x \rangle_{L_g}) \right) \quad (4)$$

where $0 \le j < N_{d'}$, $0 \le j_i < N_{e^i}, 0 \le i < k-1$.

Theorem 5. *Let* $r_{d'} = r_{e^0} \cdots r_{e^{k-2}} r_g$, $T_{d'} = T_{e^0} T_{e^1} \cdots T_{e^{k-2}} T_g$ *and* $Z_{d'} = \min\{Z_{e^0}, \ Z_g\}$. *Then, the FHS set* D' *formed in Design 2 is an* $(L_{d'}, N_{d'}, r_{d'}, W_{d'}, Z_{d'}, \left\lceil \frac{W_{d'}}{T_{d'}} \right\rceil)$ *LHZ-FHS set.*

Proof. Similar to the proof of Theorem 4 and the discussions in [7], we can easily have the conclusion of Theorem 5. □

Remark 6. It is remarkable that the Generalized method 3 in [7] is just the special case of our new Design 1 for $T_{e^i} = L_{e^i}, 0 \le i < k-1$ and $Z_{e^j} = L_{e^j} - 1, 0 \le j < k$, and the Generalized method 2 in [7] is just the special case of our new Design 2 for $T_{e^i} = L_{e^i}, 0 \le i < k-1$, $Z_{e^j} = L_{e^j} - 1, 0 \le j < k-1$ and $Z_g = L_g - 1$.

4 LHZ-FHS sets with optimal PPHC properties

Up to now, there have been few desirable FHS sets can be used in our new designs above. In this section, we will choose Liu's FHS sets in [10], Solomon's FHS sets in [11], Bao's FHS sets in [12] as base FHS sets, thus having three new classes of LHZ-FHS sets with optimal PPHC properties.

Class 1. Let p_1, p_2 be two different prime numbers and n_1, n_2 be two positive integers satisfying $\gcd(p_1^{n_1}-1, p_2^{n_2}-1)=1$, $p_1^{n_1}<p_2^{n_2}$, $n_1\geq 2$ and $n_2\geq 2$. Choose two Liu's FHS sets in [10] with parameters $(p_1(p_1^{n_1}-1), p_1^{n_1}, p_1^{n_1}, W_1, p_1^{n_1}-2, \lceil \frac{W_1}{(p_1^{n_1}-1)} \rceil)$ and $(p_2(p_2^{n_2}-1), p_2^{n_2}, p_2^{n_2}, W_2, p_2^{n_2}-2, \lceil \frac{W_2}{(p_2^{n_2}-1)} \rceil)$ as the base FHS sets E^0 and E^1 in Design 1 respectively. Then we can obtain a new FHS set D^1 defined as (3).

Theorem 7. *Let* $L_{D^1}=\mathrm{lcm}(p_1(p_1^{n_1}-1), p_2(p_2^{n_2}-1))$, *then the set* D^1 *in Class 1 is an* $\left(L_{D^1}, p_1^{n_1}p_2^{n_2}, p_1^{n_1}p_2^{n_2}, W_{D^1}, p_1^{n_1}-2, \left\lceil \frac{W_{D^1}}{(p_1^{n_1}-1)(p_2^{n_2}-1)} \right\rceil\right)$ *LHZ-FHS set. Let* $\delta_{D^1}=\gcd(p_1(p_1^{n_1}-1), p_2(p_2^{n_2}-1))$. *Then* D^1 *has optimal PPHC property if* $\delta_{D^1}>\frac{p_1p_2(p_1^{n_1}(p_1^{n_1}+2p_2^{n_2}-3)-2p_2^{n_2}+1)}{p_1^{2n_1}p_2^{n_2}-p_1^{n_1}p_2^{n_2}-1}$.

Proof. According to Theorem 1, D^1 is an $\left(L_{D^1}, p_1^{n_1}p_2^{n_2}, p_1^{n_1}p_2^{n_2}, W_{D^1}, p_1^{n_1}-2, \left\lceil \frac{W_{D^1}}{(p_1^{n_1}-1)(p_2^{n_2}-1)} \right\rceil\right)$ LHZ-FHS set. Thus, all we have left is to check its PPHC optimality. Put the parameters of D^1 into the bound (2), we have

$$\begin{aligned} H_{pzm}(D^1; W_{D^1}) &\geq \left\lceil \left\lceil \frac{((p_1^{n_1}-1)p_1^{n_1}p_2^{n_2}-p_1^{n_1}p_2^{n_2})L_{D^1}}{((p_1^{n_1}-1)p_1^{n_1}p_2^{n_2}-1)p_1^{n_1}p_2^{n_2}} \right\rceil \cdot \frac{W_{D^1}}{L_{D^1}} \right\rceil \\ &= \left\lceil \left\lceil \frac{p_1p_2}{\delta_{D^1}}\left(1-\frac{p_1^{n_1}(p_1^{n_1}+2p_2^{n_2}-3)-2p_2^{n_2}+1}{p_1^{2n_1}p_2^{n_2}-p_1^{n_1}p_2^{n_2}-1}\right) \right\rceil \cdot \frac{W_{D^1}}{L_{D^1}} \right\rceil. \end{aligned} \tag{5}$$

Therefore, if $\delta_{D^1}>\frac{p_1p_2(p_1^{n_1}(p_1^{n_1}+2p_2^{n_2}-3)-2p_2^{n_2}+1)}{p_1^{2n_1}p_2^{n_2}-p_1^{n_1}p_2^{n_2}-1}$, the bound (5) becomes

$$\begin{aligned} H_{pzm}(D^1; W_{D^1}) &\geq \left\lceil \frac{p_1p_2}{\delta_{D^1}} \cdot \frac{W_{D^1}}{L_{D^1}} \right\rceil \\ &= \left\lceil \frac{W_{D^1}}{(p_1^{n_1}-1)(p_2^{n_2}-1)} \right\rceil. \end{aligned}$$

According to Definition 3, D^1 has optimal PPHC property. □

Class 2. Let p be a prime number, n be a positive integer and q be a prime power satisfying $\gcd(p^n-1, q-1)=1$. Choose a Liu's FHS set in [10] and a Solomon's FHS set [11] with parameters $(p(p^n-1), p^n, p^n, W_1, p^n-2, \lceil \frac{W_1}{(p^n-1)} \rceil)$ and $(q-1, q, q, W_2, \lceil \frac{W_2}{q-1} \rceil)$ as the base FHS sets E^0 and E^1 in Design 1 respectively. Then we can obtain a new FHS set D^2 defined as (3).

Theorem 8. *The set D^2 in Class 2 is an $\left(L_{D^2}, p^nq, p^nq, W_{D^3}, Z_{D^2}, \left\lceil \frac{W_{D^2}}{(p^n-1)(q-1)} \right\rceil\right)$ LHZ-FHS set, where $L_{D^2} = \mathrm{lcm}(p(p^n-1), q-1)$ and $Z_{D^2} = \min\{p^n-2, q-2\}$. Let $\delta_{D^2} = \gcd(p(p^n-1), q-1)$. Then D^2 has optimal PPHC property if $\delta_{D^2} > \frac{p(2p^nq-2p^n+q^2-3q+1)}{p^nq^2-p^nq-1}$ when $p^n > q$.*

Proof. According to Theorem 1, D^2 is an $\left(L_{D^2}, p^nq, p^nq, W_{D^2}, Z_{D^2}, \left\lceil \frac{W_{D^2}}{(p^n-1)(q-1)} \right\rceil\right)$ LHZ-FHS set. Put the parameters of D^2 into the bound (2). We have

$$H_{pzm}(D^2; W_{D^2}) \geq \left\lceil \left\lceil \frac{((Z_{D^2}+1)p^nq - p^nq)L_{D^1}}{((Z_{D^2}+1)p^nq-1)p^nq} \right\rceil \cdot \frac{W_{D^2}}{L_{D^2}} \right\rceil. \tag{6}$$

When $p^n < q$, namely $Z_{D^2} = p^n - 2$, then the bound (6) becomes

$$H_{pzm}(D^2; W_{D^2}) \geq \left\lceil \left\lceil \frac{p}{\delta_{D^2}}(1 - \frac{p^{2n}+2p^nq-3p^n-2q+1}{p^{2n}q-p^nq-1}) \right\rceil \cdot \frac{W_{D^2}}{L_{D^2}} \right\rceil \tag{7}$$

As $0 < \frac{p(p^{2n}+2p^nq-3p^n-2q+1)}{\delta_{D^2}(p^{2n}q-p^nq-1)} < 1$, the bound (7) becomes

$$H_{pzm}(D^2; W_{D^2}) \quad \geq \quad \left\lceil \frac{p}{\delta_{D^2}} \cdot \frac{W_{D^2}}{L_{D^2}} \right\rceil = \left\lceil \frac{W_{D^2}}{(p^n-1)(q-1)} \right\rceil.$$

According to Definition 3, D^2 has optimal PPHC property if $p^n < q$. Similarly, when $p^n > q$, namely $Z_{D^2} = q-2$, the bound (6) becomes

$$H_{pzm}(D^2; W_{D^2}) \geq \left\lceil \left\lceil \frac{p}{\delta_{D^2}}(1 - \frac{2p^nq-2p^n+q^2-3q+1}{p^nq^2-p^nq-1}) \right\rceil \cdot \frac{W_{D^2}}{L_{D^2}} \right\rceil \tag{8}$$

Thus, if $\delta_{D^2} > \frac{p(2p^nq-2p^n+q^2-3q+1)}{p^nq^2-p^nq-1}$, the bound (8) also becomes

$$H_{pzm}(D^2; W_{D^2}) \quad \geq \quad \left\lceil \frac{p}{\delta_{D^2}} \cdot \frac{W_{D^2}}{L_{D^2}} \right\rceil = \left\lceil \frac{W_{D^2}}{(p^n-1)(q-1)} \right\rceil,$$

which means D^2 also has optimal PPHC property. □

Class 3. Choose the $(p(p^n-1), p^n, p^n, W_1, p^n-2, \left\lceil \frac{W_1}{(p^n-1)} \right\rceil)$ Liu's FHS set in [10] as the base FHS set E^0 in Design 2, and the $(tv, f, v, W_2, \left\lceil \frac{W_2}{v} \right\rceil)$ Bao's FHS set in [12] as the base FHS set G in Design 2 respectively. Provided that $\gcd(p^n-1, v) = 1$, then we can obtain a new FHS set $D^{1'}$ defined as (4).

Theorem 9. *The set $D^{1'}$ in Class 3 is an $\left(L_{D^{1'}}, p^n, p^nv, W_{D^{1'}}, Z_{D^{1'}}, \left\lceil \frac{W_{D^{1'}}}{(p^n-1)v} \right\rceil\right)$ LHZ-FHS set, where $L_{D^{1'}} = \mathrm{lcm}(p(p^n-1), tv)$ and $Z_{D^{1'}} = \min\{p^n-2, tv-1\}$. Let $\delta_{D^{1'}} = \gcd(p(p^n-1), tv)$. Then $D^{1'}$ has optimal PPHC property as long as $\delta_{D^{1'}} > \frac{pt(vp^n+p^n-v-2)}{p^{2n}-p^n-1}$ if $p^n-1 < tv$, or $\delta_{D^{1'}} > \frac{pt(tv+vp^n-v-1)}{tvp^n-1}$ otherwise.*

Proof. According to Theorem 2, $D^{1'}$ is an $\left(L_{D^{1'}}, p^n, p^n v, W_{D^{1'}}, Z_{D^{1'}}, \left\lceil \frac{W_{D^{1'}}}{(p^n-1)v} \right\rceil\right)$ LHZ-FHS set. Put the parameters of $D^{1'}$ into the bound (2). We have

$$H_{pzm}(D^{1'}; W_{D^{1'}}) \geq \left\lceil \left\lceil \frac{((Z_{D^{1'}}+1)p^n - p^n v)L_{D^{1'}}}{((Z_{D^{1'}}+1)p^n - 1)p^n v} \right\rceil \cdot \frac{W_{D^{1'}}}{L_{D^{1'}}} \right\rceil. \tag{9}$$

When $p^n - 1 < tv$, namely $Z_{D^{1'}} = p^n - 2$, then the bound (9) becomes

$$H_{pzm}(D^{1'}; W_{D^{1'}}) \geq \left\lceil \left\lceil \frac{pt}{\delta_{D^{1'}}}(1 - \frac{vp^n + p^n - v - 2}{p^{2n} - p^n - 1}) \right\rceil \cdot \frac{W_{D^{1'}}}{L_{D^{1'}}} \right\rceil \tag{10}$$

Therefore, if $\delta_{D^{1'}} > \frac{pt(vp^n+p^n-v-2)}{p^{2n}-p^n-1}$, the bound (10) becomes

$$H_{pzm}(D^{1'}; W_{D^{1'}}) \ \geq \ \left\lceil \frac{pt}{\delta_{D^{1'}}} \cdot \frac{W_{D^{1'}}}{L_{D^{1'}}} \right\rceil = \left\lceil \frac{W_{D^{1'}}}{(p^n-1)v} \right\rceil.$$

According to Definition 3, $D^{1'}$ has optimal PPHC property. Similarly, when $p^n - 1 < tv$, namely $Z_{D^{1'}} = tv - 1$, then the bound (9) becomes

$$H_{pzm}(D^{1'}; W_{D^{1'}}) \geq \left\lceil \left\lceil \frac{pt}{\delta_{D^{1'}}}(1 - \frac{tv + vp^n - v - 1}{tvp^n - 1}) \right\rceil \cdot \frac{W_{D^{1'}}}{L_{D^{1'}}} \right\rceil \tag{11}$$

Thus, if $\delta_{D^{1'}} > \frac{pt(tv+vp^n-v-1)}{tvp^n-1}$, the bound (11) also becomes

$$H_{pzm}(D^{1'}; W_{D^{1'}}) \ \geq \ \left\lceil \frac{pt}{\delta_{D^{1'}}} \cdot \frac{W_{D^{1'}}}{L_{D^{1'}}} \right\rceil = \left\lceil \frac{W_{D^{1'}}}{(p^n-1)v} \right\rceil,$$

which means $D^{1'}$ also has optimal PPHC property. □

Example 10. Let $p = 5$, $n = 3$, and α be the primitive element of the finite field $\mathbf{F}_{5^3}$ with $\alpha^3 = 2\alpha + 3$. For simplicity, we denote $\alpha^i, 0 \leq i \leq 123$ as i and 0 as 124. Then, we can have a $(620, 125, 125, W_0, 123, \left\lceil \frac{W_0}{124} \right\rceil)$ Liu's FHS set E^0 as:

$$\begin{aligned}
e_0^0 &= [0, 103, 56, 107, 22, 5, 77, 85, 67, 2, 10, 23, 79, 56, 119, 15, 24, 74, 20, 5, 20, 16,\\
&\quad \cdots, 88, 67, 109, 20, 12, 98, 111, 114, 47, 28, 109, 92, 119, 82, 73, 85, 95],\\
e_1^0 &= [93, 119, 30, 72, 4, 19, 50, 73, 108, 9, 45, 59, 111, 30, 14, 68, 66, 51, 58, 19, 58,\\
&\quad \cdots, 94, 108, 53, 58, 27, 39, 113, 35, 97, 104, 53, 118, 14, 48, 122, 73, 123],\\
&\vdots\\
e_{124}^0 &= [102, 37, 40, 7, 58, 76, 82, 77, 97, 87, 22, 65, 119, 40, 81, 23, 94, 91, 15, 76, 15,\\
&\quad \cdots, 66, 75, 73, 97, 19, 15, 1, 69, 14, 46, 104, 27, 19, 30, 81, 21, 50, 77, 75].
\end{aligned}$$

Table 1: The parameters of some known LHZ-FHS sets with optimal PPHC property and our new ones

Parameters $(L, N, r, W, L_{pz}, H_{pzm}(S;W))$	Constrains	Reference
$(k_1L_1, M_1N_1, r_1, W_1, Z_1 - 1, \left\lceil \frac{W_1}{T_1} \right\rceil)$	$M_1Z_1 = L_1$, $\gcd(Z_1+1, T_1) = 1$, $k_1(Z_1+1) \equiv 1(\mod L_1)$, $k_1 \equiv 1(\mod Z_1)$, $\mu \geq 1$	[13]
$(L_2L_3, p_2, p_1p_2, W_4, \min\{L_2, L_3\} - 1, \left\lceil \frac{W_4}{T_2L_3} \right\rceil)$	$\gcd(L_2, L_3) = 1$, $p_1(\frac{L_2}{T_2} - 1 + \eta)(\min\{L_2, L_3\}p_2 - 1) = L_2L_3(\min\{L_2, L_3\} - p_1)$, $0 < \eta \leq 1$	[9]
$\left(\mathrm{lcm}(q-1, tv), q, qv, W_5, \min\{q-1, tv\} - 1, \left\lceil \frac{W_5}{v(q-1)} \right\rceil\right)$	$\gcd(q-1, v) = 1$, $\gcd(q-1, t) > 1$, $tv < q-1$, or $tv > q-1$ and $t < \frac{l_{d^2}(q^2-q-1)}{qv-v-2}$	[7]
$\left(\mathrm{lcm}(q-1, p(p^n-1)), qp^{n-1}, qp^n, W_6, \min\{q-1, p(p^n-1)\} - 1, \left\lceil \frac{W_6}{(p^n-1)(q-1)} \right\rceil\right)$	$\gcd(q-1, p^n-1) = 1$, and $\gcd(q-1, p) > \frac{p^n(p+1)(q-1)+q(q-2-p)+1}{(q^2p^{n-1}-qp^{n-1}-1)}$ if $q-1 < p(p^n-1)$	[7]
$(p_1p_2(p_2^m-1), p_1p_2^{m-1}, p_1p_2^m, W, \min\{p_1, p_2(p_2^m-1)\} - 1, \left\lceil \frac{W}{p_1(p_2^m-1)} \right\rceil)$	$\gcd(p_1, p_2(p_2^m-1)) = 1$ and $p_1p_2^{m+1} + p_1^2 - p_1p_2 - p_2 - p_1^2p_2^{m-1} + 1 < 0$ if $p_1 = \min\{p_1, p_2(p_2^m-1)\}$	[8]
$\left(\mathrm{lcm}(p_1(p_1^{n_1}-1), p_2(p_2^{n_2}-1)), p_1^{n_1}p_2^{n_2}, p_1^{n_1}p_2^{n_2}, W_{D^1}, \min\{p_1^{n_1}-2, p_2^{n_2}-2\}, \left\lceil \frac{W_{D^1}}{(p_1^{n_1}-1)(p_2^{n_2}-1)} \right\rceil\right)$	$\gcd(p_1^{n_1}-1, p_2^{n_2}-1) = 1$, $n_1, n_2 \geq 2$, $p_1^{n_1} < p_2^{n_2}$, $\delta_{D^1} > \frac{p_1p_2(p_1^{n_1}(p_1^{n_1}+2p_2^{n_2}-3)-2p_2^{n_2}+1)}{p_1^{2n_1}p_2^{n_2}-p_1^{n_1}p_2^{n_2}-1}$	Theorem 7
$\left(\mathrm{lcm}(p(p^n-1), q-1), p^nq, p^nq, W_{D^2}, \min\{p^n-2, q-2\}, \left\lceil \frac{W_{D^2}}{(p^n-1)(q-1)} \right\rceil\right)$	$\gcd(p^n-1, q-1) = 1$, and $\delta_{D^2} > \frac{p(2p^nq-2p^n+q^2-3q+1)}{p^nq^2-p^nq-1}$ if $p^n > q$	Theorem 8
$\left(\mathrm{lcm}(p(p^n-1), tv), p^n, p^nv, W_{D^{1'}}, \min\{p^n-2, tv-1\}, \left\lceil \frac{W_{D^{1'}}}{(p^n-1)v} \right\rceil\right)$	$\gcd(p^n-1, v) = 1$, $\delta_{D^{1'}} > \frac{pt(vp^n+p^n-v-2)}{p^{2n}-p^n-1}$ if $p^n-1 < tv$, or $\delta_{D^{1'}} > \frac{pt(tv+vp^n-v-1)}{tvp^n-1}$ if $p^n-1 > tv$	Theorem 9

Let $v = 11 = 10 \times 1 + 1, t = 10, f = 1$, and 2 is a primitive root modulo v. Then, we can have a $(110, 1, 11, W_2, \left\lceil \frac{W_2}{11} \right\rceil)$ Bao's FHS G as:

$$
\begin{aligned}
g \;=\; & [0,2,8,2,9,6,10,5,2,10,10,0,4,5,4,7,1,9,10,4,9,9,0,8,10,8,3,2,7,9,8,\\
& \cdots,3,8,9,4,2,3,4,4,0,6,2,6,5,7,8,4,6,8,8,0,1,4,1,10,3,5,8,1,5,5].
\end{aligned}
$$

For simplicity, we denote $(x, y), 0 \leq x < 125, 0 \leq y < 11$ as $x * 11 + y$. Based on Class 3, we can have a new FHS set $D^{1'}$ as:

$$\begin{aligned} d_0^{1'} &= [0, 1135, 624, 1179, 251, 61, 857, 940, 739, 32, 120, 253, 873, 621, 1313, 172, \\ &\quad \cdots, 1086, 1221, 1255, 521, 309, 1209, 1015, 1314, 910, 804, 940, 1050], \\ d_1^{1'} &= [1023, 1311, 338, 794, 53, 215, 560, 808, 1190, 109, 505, 649, 1225, 335, 158, \\ &\quad \cdots, 437, 1243, 386, 1071, 1145, 593, 1301, 159, 536, 1343, 808, 1358], \\ &\vdots \\ d_{124}^{1'} &= [77, 860, 184, 563, 713, 1315, 1209, 467, 640, 934, 285, 1023, 719, 181, 917, \\ &\quad \cdots, 171, 19, 767, 154, 507, 1148, 298, 219, 333, 896, 239, 551, 852, 830]. \end{aligned}$$

By computer experiment, it can be verified that for any $0 < \tau \leq 109$ when $0 \leq i = j < 125$, and for any $0 \leq \tau \leq 109$ when $0 \leq i \neq j < 125$, the PPHC function between $d_i^{1'}$ and $d_j^{1'}$ within the correlation window length $W_{D^{1'}}, 1 \leq W_{D^{1'}} \leq 6820$ starting at $w, 0 \leq w < 6820$ can be given by

$$H_{d_i^{1'}, d_j^{1'}}(w|W_{D^{1'}}; \tau) \leq \left\lceil \frac{W_{D^{1'}}}{1364} \right\rceil.$$

Therefore, $D^{1'}$ is a $(6820, 125, 1375, W_{D_{1'}}, 109, \left\lceil \frac{W_{D^{1'}}}{1364} \right\rceil)$ LHZ-FHS set. Besides, according to Theorem 9, $D^{1'}$ has optimal PPHC property.

5 Conclusions

In this paper, through Cartesian product, we first presented two designs to construct LHZ-FHS sets. It can be verified that these two designs can include the Generalized method 2 and the Generalized method 3 in [7] as special cases. Then based on desirable known FHS sets, we obtained three new classes of LHZ-FHS sets with optimal PPHC properties. It turns out that our new LHZ-FHS sets possess new parameters not included in the relevant literature (see Table 1). It should be noted that, in Table 1, the FHS set in [13] was obtained by using interleaving technique via an optimal $(L_1, N_1, r_1, W_1, \left\lceil \frac{W_1}{T_1} \right\rceil)$ FHS set; and the FHS set in [9] was acquired via Cartesian product of two FHS sets with parameters $(L_2, N_2, p_1, W_2, \left\lceil \frac{W_2}{T_2} \right\rceil)$ and $(L_3, p_2, p_3, W_3, \left\lceil \frac{W_3}{L_3} \right\rceil)$ respectively.

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