NEW CONSTANT DIMENSION SUBSPACE CODES FROM PARALLEL LINKAGE CONSTRUCTION AND MULTILEVEL CONSTRUCTION

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ABSTRACT. A basic problem about a constant dimension subspace code is to find its maximal possible size $A_q(n, d, k)$. This paper investigates constant dimension codes with parallel linkage construction and multilevel construction and obtains new lower bounds of sizes of $A_q(18, 6, 9)$. Then we combined the Johnson type bound and got better lower bounds of $A_q(17, 6, 8)$. These low bounds are larger than previously best known bounds in [16].

1. INTRODUCTION

Let $\mathbb{F}_q$ be a finite field with q elements. The set of all subspaces of $\mathbb{F}_{q^n}$ is denoted by $\mathcal{P}_q(n)$. For any $\mathcal{U}, \mathcal{V} \in \mathcal{P}_q(n)$, the subspace distance of $\mathcal{U}$ and $\mathcal{V}$ is defined as:

$$d_S(\mathcal{U}, \mathcal{V}) = \dim(\mathcal{U} + \mathcal{V}) - \dim(\mathcal{U} \cap \mathcal{V}) = \dim(\mathcal{U}) + \dim(\mathcal{V}) - 2\dim(\mathcal{U} \cap \mathcal{V}),$$

where $\dim(\cdot)$ denotes the dimension of a vector space over $\mathbb{F}_q$. Let $\mathcal{G}_q(n, k)$ be the set of all k-dimensional $\mathbb{F}_q$-subspaces of $\mathbb{F}_{q^n}$. A subset $\mathcal{C} \subseteq \mathcal{G}_q(n, k)$ with the subspace distance d_S is called a constant dimension code. The minimum subspace distance of $\mathcal{C}$ is defined as:

$$d_S(\mathcal{C}) = \min\{d_S(\mathcal{U}, \mathcal{V}) | \mathcal{U}, \mathcal{V} \in \mathcal{C}, \mathcal{U} \neq \mathcal{V}\}.$$

$\mathcal{C}$ is called an $(n, M, d, k)_q$, if $|\mathcal{C}| = M$, and $d_S(\mathcal{C}) = d$. A basic problem about a constant dimension subspace code is to find the maximal possible size $A_q(n, d, k)$ of an $(n, M, d, k)_q$ constant dimension code.

Constant dimension subspace codes are tool for studying random network coding. It is very useful with efficient encoding and decoding algorithms. We can find various results on the construction of constant dimension codes in the literature (see [3, 4, 9, 10, 14, 15, 16, 17, 18, 23, 24]).

(1) Heide Gluesing-Luerssen and Carolyn Troha in [13] linked two constant dimension codes with small length and obtained constant dimension codes with large lower bounds. This construction was named as the linkage construction.

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(2) Liqin Xu and Hao Chen in [25] presented an interesting construction, which was named as the parallel construction, to establish new lower bounds for $A_q(2k, 2d, k)$, where $k \geq 2d$.

(3) Hao Chen et al in [5] improved the linkage construction and presented a new construction which was named as the parallel linkage construction. Many new lower bounds on $A_q(n, d, k)$ were proved from the parallel linkage construction.

(4) Fagang Li in [19] gave another effective improvement of the linkage construction which was named as multilevel linkage construction. He combined the multilevel construction and linkage construction and obtained some new lower bounds of constant dimension codes for small parameters.

(5) Shuangqing Liu, Yanxun Chang and Tao Feng in [22] combined the parallel construction and the multilevel construction and give many constant dimension codes with larger size than the previously best known codes.

In this paper, inspired by the results in [5, 22], we use the parallel linkage construction and multilevel construction to present an improved version. As a consequence, better constant dimension codes are constructed. This paper is organized as follows. In Section 2, we give some basic results. In Section 3, we show the main results. The maximal possible size $A_q(n, d, k)$ of constant dimension codes are larger than those in [16]. In Section 4, we conclude the paper.

2. Preliminaries

In this section, we will review some definitions and previous results. For more details, readers can refer to [2, 5, 6, 7, 9, 12].

2.1. MRD codes and Ferrers diagram rank-metric codes.

Definition 2.1. The rank-metric code $\mathcal{M}$ is a subset of the matrix space $\mathbb{F}_q^{m\times n}$ endowed with the rank metric $d_R(A, B) = \text{rank}(A - B)$. A linear rank-metric code $[m \times n, \rho, d]_q$ is a linear subspace of $\mathbb{F}_q^{m\times n}$ with cardinality q^ρ and distance d.

It is well-known that the number of codewords in $\mathcal{M}$ is upper bounded by

$$q^{max\{m,n\}\times(min\{m,n\}-d+1)}$$

(see [7, 12]). A code attaining this bound is called a *maximum rank − distance* (MRD) code.

Let X be a subspace of dimension k of $\mathbb{F}_q^n$. It can be represented by a $k \times n$ generator matrix whose k rows form a basis for X. There is exactly one such matrix in reduced row echelon form and it is denoted by $E(X)$.

Definition 2.2. [9] For each subspace X of dimension k of $\mathbb{F}_q^n$, let $E(X)$ be the reduced row echelon form of X. A binary row vector $v(X)$ of length n and weight k is called the *identifying vector* of X, where the k ones of $v(X)$ are exactly the pivots of $E(X)$.

Example 2.3. *Let X be a 3-dimension subspace of $\mathbb{F}_2^7$ with the following generator matrix in reduced row echelon form:*

$$E(X) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Then the identifying vector of X is $v(X) = (1101000)$.

Let $\mathcal{F}(X)$ be obtained by remove the columns which contain the pivots of $E(X)$ and the zeroes from each row of $E(X)$ to the left of the pivots. The Ferrers diagram of a k-dimension subspace X of $\mathbb{F}_q^n$, denoted by $\mathcal{F}_X$, is acquired from $\mathcal{F}(X)$ by replacing the entries of $\mathcal{F}(X)$ with dots.

Example 2.4. *Consider the 3-dimension subspace X of Example 2.3, the corresponding $\mathcal{F}(X)$ is*

$$\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ & 0 & 1 & 1 \end{array}.$$

The corresponding Ferrers diagram $\mathcal{F}_X$ of X is

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet \end{array}.$$

Definition 2.5. [2] Let $\mathcal{F}$ be a $k \times (n-k)$ Ferrers diagram. An $\mathbb{F}_q$-linear rank-metric code $\mathcal{C}_\mathcal{F} \subseteq \mathbb{F}_q^{k\times(n-k)}$ is called a *Ferrers diagram rank-metric code* if every matrix $M \in \mathcal{C}_\mathcal{F}$ has shape $\mathcal{F}$, that is, all entries of M not in $\mathcal{F}$ are zeroes. If $\mathcal{F}$ is a *full* $k \times (n-k)$ *Ferrers diagram* with $k \times (n-k)$ dots, then its corresponding Ferrers diagram rank-metric code is just a classical rank-metric code. A Ferrers diagram rank-metric code is called an $[\mathcal{F}, \rho, d]$-code if $\mathrm{rank}(M) \geq d$ for arbitrary nonzero codeword $M \in \mathcal{C}_\mathcal{F}$ and $\dim(\mathcal{C}_\mathcal{F}) = \rho$.

The following lemma in [9] plays an important role to obtain the desired minimum distance.

Lemma 2.6. *Let X and Y be two subspaces of $\mathcal{P}_q(n)$ with identifying vectors $v(X)$ and $v(Y)$, respectively, then*

$$d_S(X, Y) \geq d_H(v(X), v(Y)),$$

where $d_H(u, v)$ denotes the Hamming distance between u and v.

Lemma 2.7. [9] *Let $\mathcal{F}$ be a $k \times n$ Ferrers diagram and $\mathcal{C}_\mathcal{F} \subseteq \mathbb{F}_q^{k\times n}$ be a Ferrers diagram rank-metric code with minimum distance d. Then an upper bound of the cardinality of $\mathcal{C}_\mathcal{F}$ is $q^{min_i\{t_i\}}$, where t_i is the number of dots in $\mathcal{F}$ after removing the top i rows and rightmost $d-1-i$ columns for $0 \leq i \leq d-1$.*

An $[\mathcal{F},\rho,d]$-code is called an *optimal Ferrers diagram rank − metric code* if it attains the upper bound of $q^{min_i\{t_i\}}$. The existence of optimal Ferrers diagram rank-metric code has been proven.(see [2, 20, 21, 27]). The following Lemma is a criterion for check the existence of optimal Ferrers diagram rank-metric code.

Lemma 2.8. [8, 9] *Let $\mathcal{F}$ be a $k\times n$ $(k\geq n)$ Ferrers diagram and each of the rightmost $d-1$ columns of $\mathcal{F}$ has at least n dots. Then there exists an optimal $[\mathcal{F},\sum_{i=1}^{n-d+1}\epsilon_i,d]$ code for any prime power q, where ϵ_i is the number of dots in i-th column of $\mathcal{F}$.*

For a Ferrers diagram $\mathcal{F}$ of size $k\geq n$, one can transpose it to obtain a Ferrers diagram $\mathcal{F}^T$ of size $n\geq k$, where the rightmost column of $\mathcal{F}$ is the first row of $\mathcal{F}^T$ and so on. Thus if there exists an optimal Ferrers diagram rank-metric code for $\mathcal{F}$, then so does an optimal Ferrers diagram rank-metric code for $\mathcal{F}^T$.

Lemma 2.9. [22] *Let m,n,k_1,k_2 are integers with $m,n\geq k_1$ and $n\geq k_2$. Let $\mathcal{F}^{12}=(\mathcal{F}_1|\mathcal{F}_2^T)$ be a $n\times(m+k_2)$ Ferrers diagram, which $\mathcal{F}_1$ is a $k_1\times m$ Ferrers diagram and $\mathcal{F}_2$ is a $k_2\times n$ Ferrers diagram. If there exists an $[\mathcal{F}^{12},\rho,d]_q$ code, then there exists an $[\mathcal{F},\rho,d]$ code $\mathcal{D}$, where*

$$\mathcal{F}=\begin{pmatrix}\mathcal{F}_1 & \mathcal{F}_3\\ & \mathcal{F}_2\end{pmatrix}$$

satisfying that for any codeword $D\in\mathcal{D},D|_{\mathcal{F}_3}=O$, where $\mathcal{F}_3$ is a $k_1\times n$ Ferrers diagram and O is a $k_1\times n$ zero matrix.

For $m,k_2\geq k_1,n\leq k_2$, let $\mathcal{F}^{12}=(\mathcal{F}_1|\mathcal{F}_2)$ be a $k_2\times(m+n)$ Ferrers diagram. If there exists an $[\mathcal{F}^{12},\rho,d]_q$ code, then there exists an $[\mathcal{F},\rho,d]$ code. Similarly, for any m,n,k_1,k_2, we can choose suitable form of $\mathcal{F}^{12}$ and get the similar result with Lemma 2.9.

For a given $[k\times(n-k),\rho,d]$ rank-metric code $\mathcal{M}$, we can construct an $(n,q^\rho,2d,k)_q$ constant dimension code $\mathcal{C}_\mathcal{M}$ by lifting $\mathcal{M}$, i.e., $\mathcal{C}_\mathcal{M}=\{rs(I_k|M):M\in\mathcal{M}\}$, where $rs(\cdot)$ denotes the row space of a matrix. Similarly, let $\mathcal{F}$ be a $k\times(n-k)$ Ferrers diagram and $\mathcal{C}_\mathcal{F}\subseteq\mathbb{F}_q^{k\times(n-k)}$ is the corresponding Ferrers diagram rank-metric code. A lifted constant dimension code $\tilde{\mathcal{C}}_\mathcal{F}$ is defined as follows:

$$\tilde{\mathcal{C}}_\mathcal{F}=\{X\in\mathcal{G}_q(n,k):\mathcal{F}(X)\in\mathcal{C}_\mathcal{F},\mathcal{F}_X=\mathcal{F}\}.$$

As an immediate consequence from the lifting construction [9], we have the following.

Lemma 2.10. *Let $\mathcal{C}_\mathcal{F}\subseteq\mathbb{F}_q^{k\times(n-k)}$ be an $[\mathcal{F},\rho,d]$ Ferrers diagram rank-metric code, then its lifted code $\tilde{\mathcal{C}}_\mathcal{F}$ is an $(n,q^\rho,2d,k)_q$ constant dimension code.*

Etzion in [9] presented the *multilevel construction* to construct $(n,M,2d,k)_q$ constant dimension code with large cardinality. Let C be a binary constant-weight code with length n, weight k, and Hamming distance $2d$. For each codeword $c\in C$, let $\mathcal{F}_c$ be Ferrers diagram corresponding to identifying vector c. Then we construct an

$[\mathcal{F}_c, \rho, d]$ Ferrers diagram rank-metric code $\mathcal{C}_{\mathcal{F}_c}$ for the Ferrers diagram $\mathcal{F}_c$ of c. Consider the set defined by $\mathcal{C} = \bigcup_{c \in C} \tilde{\mathcal{C}}_{\mathcal{F}_c}$, where the constant dimension code $\tilde{\mathcal{C}}_{\mathcal{F}_c}$ is lifted by $\mathcal{C}_{\mathcal{F}_c}$. As an immediate consequence of Lemma 2.7 and 2.9, we obtain an $(n, M, 2d, k)_q$ constant dimension code, where $M = \sum_{c \in C} |\tilde{\mathcal{C}}_{\mathcal{F}_c}|$.

2.2. **Construction of constant dimension codes through linkage.**

In this section, we will review some basic notations and previous results about linkage construction.

Definition 2.11. [13] A set $\mathcal{M} \subseteq \mathbb{F}_q^{k \times n}$ of $k \times n$ matrices over $\mathbb{F}_q$ is called an *SC − representation* of a constant dimension code in $\mathcal{G}_q(n, k)$ if the following conditions are satisfied.

(a) For each $M \in \mathcal{M}$, $\mathrm{rank}(M) = k$.

(b) For any two distinct matrices $M_1, M_2 \in \mathcal{M}, rs(M_1) \neq rs(M_2)$.

Obviously, the corresponding constant dimension code of $\mathcal{M}$ is $\{rs(M) : M \in \mathcal{M}\}$, denoted by $\mathcal{C}(\mathcal{M})$.

Lemma 2.12. (**Parallel linkage construction**) [5] *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be SC-representation of two $(k+n, N_1, d, k)_q$ and $(n+k, N_2, d, k)_q$ constant dimension codes satisfying that each $k \times (k+n)$ matrix in $\mathcal{M}_1$ is of the form $(M_1|M_1')$, where M_1 is a non-singular $k \times k$ matrix. Suppose $d \leq k$. Let $\mathcal{Q}_1 \subseteq \mathbb{F}_q^{k \times k}$ be a code with rank distance $\frac{d}{2}$ and N_3 elements. Let $\mathcal{Q}_2 \subseteq \mathbb{F}_q^{k \times k}$ be a code with rank distance $\frac{d}{2}$ and N_4 elements such that the rank of each element in $\mathcal{Q}_2$ is at most $k - \frac{d}{2}$. Consider the set defined by $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, where*

$$\mathcal{C}_1 = \{rs(M_1|M_1'|Q) : (M_1|M_1') \in \mathcal{M}_1, Q \in \mathcal{Q}_1\},$$
$$\mathcal{C}_2 = \{rs(Q'|M_2) : M_2 \in \mathcal{M}_2, Q' \in \mathcal{Q}_2\}.$$

Then $\mathcal{C}$ is a $(k+n+k, N_1N_3 + N_2N_4, d, k)_q$ constant dimension code.

2.3. **Delsarte theorem and Jahnson type bound.**

In this section, we recall some basic notations and results about Delsarte theorem.

Definition 2.13. Let $\mathcal{M} \subseteq \mathbb{F}_q^{k \times n}$, then the *rank distribution* of $\mathcal{M}$ is defined by

$$A_i(\mathcal{M}) = |\{M \in \mathcal{M} : \mathrm{rank}(M) = i\}|.$$

The rank distribution of an MRD code is completely determined by its parameters. The following results can be referred to Corollary 26 in [6] or Theorem 5.6 in [7]. The Delsarte Theorem is essential to calculate the final results in this paper.

Lemma 2.14. (**Delsarte1978**) *Let $\mathcal{M} \subseteq \mathbb{F}_q^{k \times n} (n \geq k)$ is an MRD code with rank distance d, then its rank distribution is given by*

$$A_r(\mathcal{M}) = \binom{k}{r}_q \sum_{i=0}^{r-d} (-1)^i q^{\binom{i}{2}} \binom{r}{i}_q \left(\frac{q^{n(k-d+1)}}{q^{n(k+i-r)}} - 1\right).$$

Johnson type bound [11, 26] can be used to get some better constant constant dimension subspace codes in our construction.

Lemma 2.15. [11] *Let n, k, d be integers with $k \geq 2d$. Then*
$$A_q(n, 2d, k) \leq \frac{q^n - 1}{q^k - 1} A_q(n-1, 2d, k-1).$$

3. Main results

In this section, we combine the parallel linkage construction and the multilevel construction and obtain some new lower bounds of constant dimension codes.

Let m, n, k, d are integers with $m \geq k+d, n \geq d$ and $k \geq 3d$. Cosider the set of identifying vectors with length $m+n$ and weight k defined by
$$S = \{s_i = (\underbrace{1\dots1}_{k-3d}|\alpha_i|\beta_i|\underbrace{1\dots1}_{d}\underbrace{0\dots0}_{n-d}) : \alpha_i \in \mathbb{F}_2^{2d}, \beta_i \in \mathbb{F}_2^{m-k+d}, 0 \leq i \leq 3\},$$
where $\alpha_0 = (\underbrace{0\cdots0}_{d}\underbrace{0\cdots0}_{d}), \alpha_1 = (\underbrace{0\cdots0}_{d}\underbrace{1\cdots1}_{d}), \alpha_2 = (\underbrace{1\cdots1}_{d}\underbrace{0\cdots0}_{d}), \alpha_3 = (\underbrace{1\cdots1}_{d}\underbrace{1\cdots1}_{d})$, $w_H(\beta_0) = 2d$, $w_H(\beta_1) = w_H(\beta_2) = d$, and $w_H(\beta_3) = 0$.

It is obvious that $d_H(s_i, s_j) \geq d_H(\alpha_i, \alpha_j) + d_H(\beta_i, \beta_j) \geq 2d, 0 \leq i \neq j \leq 3$. Without loss of generality, let $\beta_0 = (\underbrace{1\dots1}_{2d}\underbrace{0\dots0}_{m-k-d})$. Note that the reduced row echelon form $E(s_0)$ coresponding to identifying vector s_0 is
$$E(s_0) = \begin{pmatrix} I_{k-3d} & \mathcal{F}^{01} & \mathcal{F}^{02} & O & O & \mathcal{F}^{03} & O & \mathcal{F}^{06} \\ O & O & O & I_d & O & \mathcal{F}^{04} & O & \mathcal{F}^{07} \\ O & O & O & O & I_d & \mathcal{F}^{05} & O & \mathcal{F}^{08} \\ O & O & O & O & O & O & I_d & \mathcal{F}^{09} \end{pmatrix},$$
where $\mathcal{F}^{01}$, $\mathcal{F}^{02}$ are $(k-3d) \times d$ full Ferrers diagram, $\mathcal{F}^{03}$ is a $(k-3d) \times (m-k-d)$ full Ferrers diagram, $\mathcal{F}^{04}$, $\mathcal{F}^{05}$ are $d \times (m-k-d)$ full Ferrers diagram, $\mathcal{F}^{06}$ is a $(k-3d) \times (n-d)$ full Ferrers diagram, $\mathcal{F}^{07}, \mathcal{F}^{08}, \mathcal{F}^{09}$ are $d \times (n-d)$ full Ferrers diagrams, and O is zero matrix with suitable size.

Let
$$\mathcal{F}^0 = \begin{pmatrix} \mathcal{F}^{01} & \mathcal{F}^{02} & \mathcal{F}^{03} & \mathcal{F}^{06} \\ & & \mathcal{F}^{04} & \mathcal{F}^{07} \\ & & \mathcal{F}^{05} & \mathcal{F}^{08} \\ & & & \mathcal{F}^{09} \end{pmatrix}$$
be a $k \times (m+n-k)$ Ferrers diagram and $\mathcal{D}_{\mathcal{F}^0}$ be a Ferrers diagram rank-metric code with shape $\mathcal{F}^0$. Consider the set of k-dimension subspaces in $\mathbb{F}_q^{m+n}$ defined by
$$\mathcal{C}_{\mathcal{F}_0} = \left\{ rs \begin{pmatrix} I_{k-3d} & \mathcal{D}^{01} & \mathcal{D}^{02} & O & O & \mathcal{F}^{03} & O & \mathcal{D}^{06} \\ O & O & O & I_d & O & \mathcal{D}^{04} & O & \mathcal{D}^{07} \\ O & O & O & O & I_d & \mathcal{D}^{05} & O & \mathcal{D}^{08} \\ O & O & O & O & O & O & I_d & \mathcal{D}^{09} \end{pmatrix} : \begin{pmatrix} \mathcal{D}^{01} & \mathcal{D}^{02} & \mathcal{D}^{03} & \mathcal{D}^{06} \\ O & O & \mathcal{D}^{04} & \mathcal{D}^{07} \\ O & O & \mathcal{D}^{05} & \mathcal{D}^{08} \\ O & O & O & \mathcal{D}^{09} \end{pmatrix} \in \mathcal{D}_{\mathcal{F}^0} \right\},$$

Take a $(k-d)\times(m+n-k)$ Ferrers diagram $\mathcal{F}^{0'}=\begin{pmatrix} \mathcal{F}^{04} & \mathcal{F}^{06} \\ & \mathcal{F}^{08} \\ & \mathcal{F}^{09} \end{pmatrix}$, by Lemma 2.8, there exists an optimal $[\mathcal{F}^{0'},(n-d)(k-2d)+m+n-k-2d,d]$ code. By Lemma 2.9, we obtain an $[\mathcal{F}^{0},(n-d)(k-2d)+m+n-k-2d,d]$ code $\mathcal{D}'_{\mathcal{F}}$. Note that each codeword in $\mathcal{D}'_{\mathcal{F}}$ is of the form $\left(\begin{array}{c|cc} & O^{02} & * \\ \mathrm{O}^{01} & * & O^{03} \\ & O^{04} & * \\ & O^{05} & * \end{array}\right)$, where O^{01} is a $d\times 2d$ zero matrix, O^{02} is a $(k-3d)\times(m-k-d)$ zero matrix, O^{04},O^{05} are two $d\times(m-k-d)$ zero matrix, and O^{03} is a $d\times(n-d)$ zero matrix.

Take $\mathcal{D}_{\mathcal{F}^0}=\mathcal{D}'_{\mathcal{F}}$, then $\mathcal{D}_{\mathcal{F}^0}$ is a Ferrers diagram rank-metric code in $\mathcal{F}^0$ with minimum rank distance d and cardinality $q^{(n-d)(k-2d)+m+n-k-2d}$. By Lemma 2.10, $\mathcal{C}_{\mathcal{F}_0}$ is an $(m+n,q^{(n-d)(k-2d)+m+n-k-2d},2d,k)_q$ constant dimension code.

The case of β_1 is similar to the case of β_2. For simplicity, we omit the case of β_1. Without loss of generality, let $\beta_2=(\underbrace{1\dots1}_{d}\underbrace{0\dots0}_{m-k})$. Note that the reduced row echelon form $E(s_2)$ coresponding to identifying vector s_2 is

$$E(s_2)=\begin{pmatrix} I_{k-3d} & O & \mathcal{F}^{21} & O & \mathcal{F}^{23} & O & \mathcal{F}^{26} \\ O & I_d & \mathcal{F}^{22} & O & \mathcal{F}^{24} & O & \mathcal{F}^{27} \\ O & O & O & I_d & \mathcal{F}^{25} & O & \mathcal{F}^{28} \\ O & O & O & O & O & I_d & \mathcal{F}^{29} \end{pmatrix},$$

where $\mathcal{F}^{21}$ is a $(k-3d)\times d$ full Ferrers diagram, $\mathcal{F}^{22}$ is a $d\times d$ full Ferrers diagram, $\mathcal{F}^{23}$ is a $(k-3d)\times(m-k)$ full Ferrers diagram, $\mathcal{F}^{24}$ and $\mathcal{F}^{25}$ are two $d\times(m-k)$ full Ferrers diagram, $\mathcal{F}^{26}$ is a $(k-3d)\times(n-d)$ full Ferrers diagram, $\mathcal{F}^{27},\mathcal{F}^{28},\mathcal{F}^{29}$ are $d\times(n-d)$ full Ferrers diagrams, and O is zero matrix with suitable size.

Let

$$\mathcal{F}^2=\begin{pmatrix} \mathcal{F}^{21} & \mathcal{F}^{23} & \mathcal{F}^{26} \\ \mathcal{F}^{22} & \mathcal{F}^{24} & \mathcal{F}^{27} \\ & \mathcal{F}^{25} & \mathcal{F}^{28} \\ & & \mathcal{F}^{29} \end{pmatrix}$$

be a $k\times(m+n-k)$ Ferrers diagram. Consider the set of k-dimension subspaces in $\mathbb{F}_q^{m+n}$ defined by

$$\mathcal{C}_{\mathcal{F}_2}=\left\{ rs\begin{pmatrix} I_{k-3d} & O & \mathcal{D}^{21} & O & \mathcal{D}^{23} & O & \mathcal{D}^{26} \\ O & I_d & \mathcal{D}^{22} & O & \mathcal{D}^{24} & O & \mathcal{D}^{27} \\ O & O & O & I_d & \mathcal{D}^{25} & O & \mathcal{D}^{28} \\ O & O & O & O & O & I_d & \mathcal{D}^{29} \end{pmatrix} : \begin{pmatrix} \mathcal{D}^{21} & \mathcal{D}^{23} & \mathcal{D}^{26} \\ \mathcal{D}^{22} & \mathcal{D}^{24} & \mathcal{D}^{27} \\ O & \mathcal{D}^{25} & \mathcal{D}^{28} \\ O & O & \mathcal{D}^{29} \end{pmatrix} \in \mathcal{D}_{\mathcal{F}^2} \right\},$$

Take a $(k-d)\times(m+n-k)$ Ferrers diagram $\mathcal{F}^{2'}=\begin{pmatrix}\mathcal{F}^{22} & \mathcal{F}^{24} & \mathcal{F}^{26}\\ & & \mathcal{F}^{28}\\ & & \mathcal{F}^{29}\end{pmatrix}$, by Lemma 2.8, there exists an optimal $[\mathcal{F}^{2'},(n-d)(k-2d)+m+n-k,d]$ code. By Lemma 2.9, we obtain an $[\mathcal{F}^2,(n-d)(k-2d)+m+n-k,d]$ code $\mathcal{D}''_{\mathcal{F}}$. Note that each codeword in $\mathcal{D}''_{\mathcal{F}}$ is of the form $\begin{pmatrix}O^{21} & O^{24} & \mathcal{D}^{26}\\ \mathcal{D}^{22} & \mathcal{D}^{24} & O^{27}\\ O^{22} & O^{25} & \mathcal{D}^{28}\\ O^{23} & O^{26} & \mathcal{D}^{29}\end{pmatrix}$, where O_{21} is a $(k-3d)\times d$ zero matrix, O_{22},O_{23} are two $d\times d$ zero matrix, O_{24} is a $(k-3d)\times(m-k)$ zero matrix, O_{25},O_{26} are two $d\times(m-k)$ zero matrix, and O_{27} is a $d\times(n-d)$ zero matrix.

Take $\mathcal{D}_{\mathcal{F}^2}=\mathcal{D}''_{\mathcal{F}}$, then $\mathcal{D}_{\mathcal{F}^2}$ is a Ferrers diagram rank-metric code in $\mathcal{F}^2$ with minimum rank distance d and cardinality $q^{(n-d)(k-2d)+m+n-k}$. By Lemma 2.10, $\mathcal{C}_{\mathcal{F}_2}$ is an $(m+n,q^{(n-d)(k-2d)+m+n-k},2d,k)_q$ constant dimension code.

Note that the reduced row echelon form $E(s_3)$ corresponding to identifying vector s_3 is

$$E(s_3)=\begin{pmatrix}I_{k-3d} & O & O & \mathcal{F}^{31} & O & \mathcal{F}^{34}\\ O & I_d & O & \mathcal{F}^{32} & O & \mathcal{F}^{35}\\ O & O & I_d & \mathcal{F}^{33} & O & \mathcal{F}^{36}\\ O & O & O & O & I_d & \mathcal{F}^{37}\end{pmatrix},$$

where $\mathcal{F}^{31}$ is a $(k-3d)\times(m-k+d)$ full Ferrers diagram, $\mathcal{F}^{32},\mathcal{F}^{33}$ are two $d\times(m-k+d)$ full Ferrers diagram, $\mathcal{F}^{34}$ is a $(k-3d)\times(m-k+d)$ full Ferrers diagram, $\mathcal{F}^{35},\mathcal{F}^{36},\mathcal{F}^{37}$ are $d\times(n-d)$ full Ferrers diagrams, and O is zero matrix with suitable size. Let

$$\mathcal{F}^3=\begin{pmatrix}\mathcal{F}^{31} & \mathcal{F}^{34}\\ \mathcal{F}^{32} & \mathcal{F}^{35}\\ \mathcal{F}^{33} & \mathcal{F}^{36}\\ & \mathcal{F}^{37}\end{pmatrix}$$

be a $k\times(m+n-k)$ Ferrers diagram. Consider the set of k-dimension subspaces in $\mathbb{F}_q^{m+n}$ defined by

$$\mathcal{C}_{\mathcal{F}_3}=\left\{rs\begin{pmatrix}I_{k-3d} & O & O & \mathcal{D}^{31} & O & \mathcal{D}^{34}\\ O & I_d & O & \mathcal{D}^{32} & O & \mathcal{D}^{35}\\ O & O & I_d & \mathcal{D}^{33} & O & \mathcal{D}^{36}\\ O & O & O & O & I_d & \mathcal{D}^{37}\end{pmatrix}:\begin{pmatrix}\mathcal{D}^{31} & \mathcal{D}^{34}\\ \mathcal{D}^{32} & \mathcal{D}^{35}\\ \mathcal{D}^{33} & \mathcal{D}^{36}\\ O & \mathcal{D}^{37}\end{pmatrix}\in\mathcal{D}_{\mathcal{F}^3}\right\},$$

Take a $(k-d)\times(m+n-k)$ Ferrers diagram $\mathcal{F}^{3'}=\begin{pmatrix}\mathcal{F}^{32} & \mathcal{F}^{34}\\ & \mathcal{F}^{36}\\ & \mathcal{F}^{37}\end{pmatrix}$, by Lemma 2.8, there exists an optimal $[\mathcal{F}^{3'},(n-d)(k-2d)+m+n-k,d]$ code. By Lemma 2.9, we obtain an $[\mathcal{F}^3,(n-d)(k-2d)+m+n-k,d]$ code $\mathcal{D}'''_{\mathcal{F}}$. Note that each codeword in $\mathcal{D}'''_{\mathcal{F}}$ is of the form $\begin{pmatrix} O^{31} & * \\ * & O^{35} \\ O^{33} & * \\ O^{34} & * \end{pmatrix}$, where O_{31} is a $(k-3d)\times(m-k+d)$ zero matrix, O_{33}, O_{34} are two $d\times(m-k+d)$ zero matrix and O_{35} is a $d\times(n-d)$ zero matrix. Take $\mathcal{D}_{\mathcal{F}^3} = \mathcal{D}'''_{\mathcal{F}}$, then $\mathcal{D}_{\mathcal{F}^3}$ is a Ferrers diagram rank-metric code in $\mathcal{F}^3$ with minimum rank distance d and cardinality $q^{(n-d)(k-2d)+m+n-k}$. By Lemma 2.10, $\mathcal{C}_{\mathcal{F}_3}$ is an $(m+n, q^{(n-d)(k-2d)+m+n-k}, 2d, k)_q$ constant dimension code.

Similar to the above, we can obtained constant dimension codes $\mathcal{C}_{\mathcal{F}_1}$ with suitable form and parameters. Consider the set of k-dimension subspaces in $\mathbb{F}_q^{m+n}$ defined by

$$\mathcal{C}_{\mathcal{F}} = \cup_{i=0}^{3}\mathcal{C}_{\mathcal{F}_i}.$$

It is clear that $\mathcal{C}_{\mathcal{F}}$ is an $(m+n, \sum_{i=0}^{3}|\mathcal{D}_{\mathcal{F}^i}|, 2d, k)_q$ constant dimension code.

Theorem 3.1. *Let $\mathcal{U}\subseteq\mathbb{F}_q^{k\times m}$ be SC-representation of constant dimension codes $\mathcal{C}(\mathcal{U})$ with $m\geq k, |\mathcal{U}| = N_1$, and $d_S(\mathcal{C}(\mathcal{U})) = 2d$. Let $\mathcal{P}\subseteq\mathbb{F}_q^{k\times n}$ be a rank-metric code with $|\mathcal{P}| = N_2$ and $d_R(\mathcal{P}) = d$. Define a constant dimension code*

$$\mathcal{C}_1 = \{rs(U|P) : U\in\mathcal{U}, P\in\mathcal{P}\}.$$

Let $\mathcal{V}\subseteq\mathbb{F}_q^{k\times n}$ be SC-representation of constant dimension codes $\mathcal{C}(\mathcal{V})$ with $n\geq k, |\mathcal{V}| = N_3$, and $d_S(\mathcal{C}(\mathcal{V})) = 2d$. Let $\mathcal{Q}\subseteq\mathbb{F}_q^{k\times n}$ be a rank-metric code with $|\mathcal{Q}| = N_4$ and $d_R(\mathcal{Q}) = d$ such that the rank of each matrix in $\mathcal{Q}$ is at most $k-d$. Define a constant dimension code

$$\mathcal{C}_2 = \{rs(Q|V) : Q\in\mathcal{Q}, V\in\mathcal{V}\}.$$

Consider the constant dimension code defined by $\mathcal{C} = \mathcal{C}_1\cup\mathcal{C}_2\cup\mathcal{C}_{\mathcal{F}}$, where $\mathcal{C}_{\mathcal{F}} = \cup_{i=0}^{3}\mathcal{C}_{\mathcal{F}_i}$ is the same as above. Then $\mathcal{C}$ is an $(m+n, |\mathcal{C}|, 2d, k)_q$ constant dimension code with $|\mathcal{C}| = N_1N_2 + N_3N_4 + \sum_{i=0}^{3}|\mathcal{D}_{\mathcal{F}^i}|$.

Proof. By Lemma 2.12, $\mathcal{C}_1$ is an $(m+n, N_1N_2, 2d, k)_q$ constant dimension code and $\mathcal{C}_2$ is an $(m+n, N_3N_4, 2d, k)_q$ constant dimension code. By the definitions of $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_{\mathcal{F}}$, $\mathcal{C}_1\cap\mathcal{C}_2\cap\mathcal{C}_{\mathcal{F}} = \emptyset$ and $|\mathcal{C}| = N_1N_2 + N_3N_4 + \sum_{i=0}^{3}|\mathcal{D}_{\mathcal{F}^i}|$.

For each subspaces $\mathcal{X} = rs(U|P)\in\mathcal{C}_1$, $\mathcal{Y} = rs(Q|V)\in\mathcal{C}_2$ and $\mathcal{Z}\in\mathcal{C}_{\mathcal{F}}$. Let $u = (u_1, 0, \ldots, 0)$, $v = (v_1, v_2)$ and $w = (w_1, w_2)$ be identifying vectors corresponding to $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, respectively, where $u_1, v_1, w_1\in\mathbb{F}_2^m$ and $v_2, w_2\in\mathbb{F}_2^n$. By the definitions of $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_{\mathcal{F}}$, then $d_H(u_1) = k$, $d\leq d_H(v_1)\leq k-d$, $d_H(v_2)\geq d$, $d_H(w_1) = k-d$ and $d_H(w_2) = d$. By Lemma 2.6, $d_S(\mathcal{X},\mathcal{Y})\geq d_H(u,v) = d_H(u_1,v_1) + d_H(u_2)\geq 2d$ and $d_S(\mathcal{X},\mathcal{Z})\geq d_H(u,w) = d_H(u_1,w_1) + d_H(w_2) = 2d$.

It suffices to examine the subspace distance of $\mathcal{Y}\in\mathcal{C}_2$ and $\mathcal{Z}\in\mathcal{C}_{\mathcal{F}}$. Without loss of generality, let $\mathcal{Z}\in\mathcal{C}_{\mathcal{F}_3}$

By the definition of subspace distance,

$$d_S(\mathcal{Y},\mathcal{Z}) = 2\,\mathrm{rank}\left(\begin{array}{cccc|cc} I_{k-3d} & O & O & O & O & \mathcal{D}^{14} \\ O & I_d & O & \mathcal{D}^{12} & O & O \\ O & O & I_d & O & O & \mathcal{D}^{16} \\ O & O & O & O & I_d & \mathcal{D}^{17} \\ \hline & & Q & & & V \end{array}\right) - 2k.$$

We can exchange columns in the last n columns to make the last k columns in V be a $k\times k$ unit matrix I_k : $V=(V'|I_K)$, V' is a matrix with $k\times(n-k)$. In the meanwhile, $\begin{pmatrix} O & \mathcal{D}^{14} \\ O & O \\ O & \mathcal{D}^{16} \\ I_d & \mathcal{D}^{17} \end{pmatrix}$ will be transformed to $\begin{pmatrix} A^{11} & A^{14} \\ O & O \\ A^{12} & A^{15} \\ A^{13} & A^{16} \end{pmatrix}$ as follow :

$$d_S(\mathcal{Y},\mathcal{Z}) = 2\,\mathrm{rank}\left(\begin{array}{cccc|cc} I_{k-3d} & O & O & O & A^{11} & A^{14} \\ O & I_d & O & \mathcal{D}^{12} & O & O \\ O & O & I_d & O & A^{12} & A^{15} \\ O & O & O & O & A^{13} & A^{16} \\ \hline & & Q & & V' & I_k \end{array}\right) - 2k.$$

It is obvious that $d_S(\mathcal{Y},\mathcal{Z}) \geq 2\Big(rank\begin{pmatrix} O & I_d & O & \mathcal{D}^{12}\end{pmatrix} + rank(I_k)\Big) - 2k = 2d.$

In summary, the minimum distance of $\mathcal{C}$ is $2d$.

This completes the proof. □

Corollary 3.2. *Let m,n,k,d be integers with $m,n\geq k$. Then*

$$A_q(m+n,2d,k) \;\; \geq \;\; A_q(m,2d,k)q^{m\times(k-d+1)} + \sum_{j=d}^{k-d} A_j(\mathcal{Q})A_q(n,2d,k) + \sum_{i=0}^{3} |\mathcal{D}_{\mathcal{F}^i}|.$$

Proof. Let $\mathcal{P}$ be a MRD code with $d_R(\mathcal{P})=d$, then $N_2=q^{m\times(k-d+1)}$. Let $\mathcal{Q}$ be a MRD code with $d_R(\mathcal{Q})=d$, then by Lemma 2.14, $N_4=\sum_{j=d}^{k-d}A_j(\mathcal{Q})$. It is clear that the maximal size of $\mathcal{C}_1$ and $\mathcal{C}_2$ are $A_q(m,2d,k)q^{m\times(k-d+1)}$ and $\sum_{j=d}^{k-d}A_j(\mathcal{Q})A_q(n,2d,k)$, respectively. By Theorem 3.1, the conclusion holds. □

In the proof of Theorem 3.1, one can choose other identifying vectors to change the details of the multilevel construction.

Example 3.3. *Assume $m=n=k=9, d=3, q=2$. Then*

$$S=\{(111111000111000000),(111000111111000000),(000111111111000000)\}$$

is the set of identifying vectors. There exists constant dimension codes

$$\mathcal{C}_{\mathcal{F}_1} = \left\{ rs\begin{pmatrix} I_3 & O & \mathcal{D}^{11} & O & O \\ O & I_3 & O & O & \mathcal{D}^{12} \\ O & O & O & I_3 & \mathcal{D}^{13} \end{pmatrix} : \begin{pmatrix} \mathcal{D}^{11} & O \\ O & \mathcal{D}^{13} \\ O & \mathcal{D}^{13} \end{pmatrix} \in \mathcal{D}_{\mathcal{F}^1} \right\},$$

$$\mathcal{C}_{\mathcal{F}_2}=\left\{rs\begin{pmatrix} I_3 & \mathcal{D}^{21} & O & O & O\\ O & O & I_3 & O & \mathcal{D}^{22}\\ O & O & O & I_3 & \mathcal{D}^{23}\end{pmatrix}:\begin{pmatrix}\mathcal{D}^{21} & O\\ O & \mathcal{D}^{22}\\ O & \mathcal{D}^{23}\end{pmatrix}\in\mathcal{D}_{\mathcal{F}^2}\right\},$$

$$\mathcal{C}_{\mathcal{F}_3}=\left\{rs\begin{pmatrix} O & I_3 & O & O & O\\ O & O & I_3 & O & \mathcal{D}^{53}\\ O & O & O & I_3 & \mathcal{D}^{54}\end{pmatrix}:\begin{pmatrix}\mathcal{D}^{53}\\ \mathcal{D}^{54}\end{pmatrix}\in\mathcal{D}_{\mathcal{F}^3}\right\},$$

where $\mathcal{C}_{\mathcal{F}_1}$ *is a* $(18,2^{27},6,9)$ *lifted Ferrers diagram rank-metric code,* $\mathcal{C}_{\mathcal{F}_2}$ *is a* $(18,2^{27},6,9)$ *lifted Ferrers diagram rank-metric code, and* $\mathcal{C}_{\mathcal{F}_3}$ *is a* $(18,2^{24},6,9)$ *lifted Ferrers diagram rank-metric code by Lemma 2.8, Lemma 2.9 and Lemma 2.10. Then* $|\mathcal{C}_{\mathcal{F}}|=|\mathcal{C}_{\mathcal{F}_1}|+|\mathcal{C}_{\mathcal{F}_2}|+|\mathcal{C}_{\mathcal{F}_3}|=2\times 2^{27}+2^{24}=285212672$*. By Lemma 2.14,* $N_4=\sum_{j=3}^{6}A_j(\mathcal{Q})=48173119697085439$*. It is clear that the maximal size of* $\mathcal{C}_1\cup\mathcal{C}_2$ *is* $A_2(9,6,9)\times 2^{63}+A_2(9,6,9)\times\sum_{j=3}^{6}A_j(\mathcal{Q})$*. Then* $A_2(18,6,9)\geq 1\times 2^{63}+1\times 48173119697085439+285212672=9271545156837073919$*, which exceeds the current best bound* $A_2(18,6,9)=9271545156585415680$ *in* [16].

Corollary 3.4. *Let* $m=n=k=9$ *and* $d=3$*. Then*

$$A_q(18,6,9)\ \geq\ A_q(9,6,9)\times q^{63}+A_q(9,6,9)\times\sum_{j=3}^{6}A_j(\mathcal{Q})+2\times q^{27}+q^{24}.$$

Corollary 3.5. *Combining with the Johnson type bound in Lemma 2.15, then*

$$A_q(17,6,8)\ \geq\ \frac{1}{q^n+1}\Big(q^{63}+\sum_{j=3}^{6}A_j(\mathcal{Q})+2\times q^{27}+q^{24}\Big).$$

Example 3.6. *Let* $q=2$*, then* $A_2(17,6,8)=\frac{1}{2^9+1}(2^{63}+\sum_{j=3}^{6}A_j(\mathcal{Q})+2\times 2^{27}+2^{24})=\frac{1}{2^9+1}\times 9271545156837073919\geq 18073187440228214$*, which exceeds the current best bound* 18073187439737653 *in* [16].

4. Conclusion

In this paper, we combined parallel linkage construction and multilevel construction and obtained new lower bounds of the sizes of constant dimension codes. By using Corollary 3.4 and the data in the website (http://subspacecodes.unibayreuth.de) which provide tables for the presently best known lower and upper bounds for subspace codes, we have improved the lower bounds of $A_q(18,6,9)$ (listed in Table 2). Then we combined the Johnson type bound and got better lower bounds of $A_q(17,6,8)$.

q	Corollary 3.4	[16]
2	9271545156837073919	9271545156585415680
3	1144661280188113244282469293295	1144661280188113229313703859802
4	850710581461828032765396613918 44343808	850710581461828032765039140698 02090496
5	108420289965710977906690846620 4619648291015632	108420289965710977906690845136 3063099218750007
7	174251503388975551318884922599 501083078777442606195515	174251503388975551318884922599 369849935281754612330830
8	784637723721919791138381634635 240574256113712612612505600	784637723721919791138381634635 235743275201736965558894592
9	131002051249386633920687030232 9188829727611866588380431671613	131002051249386633920687030232 9188713507904303585133560754636

Table 1. $A_q(18, 6, 9)$

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